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ON EQUIVALENCE OF STRONG AND WEAK CONVERGENCE IN L_1-SPACES UNDER EXTREME POINT CONDITIONS

BY

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Dedicated to Lucien LeCam, on the occasion of his 65th birthday

ABSTRACT

Under suitable extreme point conditions weak convergence can imply strong convergence in L_1-spaces [28, 31, 12, 26] Here a number of such results are generalized by means of a unifying, very general approach using Young measures. The required results from Young measure theory are derived in a new fashion, based on pointwise averages [6], from well-known results on weak convergence of probability measures.

1. Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. The following basic result for integrable real-valued functions is well-known in measure theory [15, Thm.II.26].

THEOREM 1.1: *Let (u_k) be a sequence of integrable functions $u_k : \Omega \to \mathbf{R}$ such that*

$$u_k \to u_0 \quad \textit{weakly in } \mathcal{L}^1_{\mathbf{R}}(\mu).$$

Suppose that

$$u_0(\omega) \leq \liminf_{k\to\infty} u_k(\omega) \text{ a.e.}$$

Then there is also strong convergence in $\mathcal{L}^1_{\mathbf{R}}(\mu)$:

$$\lim_{k\to\infty} \int_\Omega |u_k - u_0| d\mu = 0.$$

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The interpretation of this result is that the inequality involving the limes inferior prevents $(u_k(\omega))$ from oscillating around $u_0(\omega)$, a fact which forces strong and weak convergence in $\mathcal{L}^1_{\mathbf{R}}(\mu)$ to coincide.

In more dimensions such oscillations can be suppressed similarly by an extreme point condition for the values $u_0(\omega)$, so as to force equivalence of strong and weak convergence. This was shown recently by Visintin [31], who gave the following result (see also [28] for related well-known work in this direction):

THEOREM 1.2: *Let (u_k) be a sequence of integrable functions $u_k : \Omega \to \mathbf{R}^d$ such that*

$$u_k \to u_0 \quad \text{weakly in } \mathcal{L}^1_{\mathbf{R}^d}(\mu).$$

Suppose that

$$u_0(\omega) \in \partial_e \text{cl co } \{u_k(\omega) : k \in \mathbf{N}\} \text{ a.e.}$$

Then

$$\lim_{k\to\infty} \int_\Omega |u_k - u_0| d\mu = 0.$$

Here $\partial_e A$ denotes the set of all extreme points of a set A; cl A denotes its closure; co A stands for its convex hull.

As shown by Visintin [31], this result can be used to obtain existence results for partial differential equations and to obtain results on well-posedness (in Tychonov's sense) for certain variational problems [31].

A refinement of Theorem 1.2 was obtained by Rzežuchowski [25], who used extremal faces and equivalent maximality with respect to lexicographical orders in $\mathbf{R}^d$; cf. [23]. A recent generalization of his result can be found in [8].

The following example, due to Visintin, shows that Theorem 1.2 is no longer valid in infinite dimensions [31, p. 445], even with constant functions u_k.

Example 1.3: Consider $E := \ell^2 \times \mathbf{R}$. Let (e_k) be the sequence of unit vectors in ℓ^2, and consider the constant functions $u_k : \Omega \to E$ given by $u_k(\omega) := (e_k, 1/k)$. Then $(e_k, 1/k) \to (0,0)$ weakly in E, but not strongly, of course. Yet $(0,0)$ is an extreme point of the set ∂_ecl co$\{u_k(\omega) : k \in \mathbf{N}\}$.

Nevertheless, it is possible to extend the result to Bochner integrable functions taking values in a reflexive Banach space E, as was shown in [3]. For this purpose a notion of *limited* convergence in $\mathcal{L}^1_E(\mu)$ was introduced, which is weaker than ordinary strong convergence if E is infinite-dimensional, but which amounts to strong convergence if E is finite-dimensional. The proof in [3], based on using the theory of Young measures [2, 4, 29], is rather more transparent than the proof in [31] (an interesting survey of these two proofs plus a number of related results was recently given by Valadier [30]). Recently, Castaing [12], following the pattern of [3], proved an extension of Visintin's result to a nonreflexive Banach space under a relative norm-compactness condition for the values $(u_k(\omega))$. He also gave a related limited convergence result. Very recently, Rzeżuchowski [26] obtained a strong convergence result for the problem considered in [3], by strengthening the extreme point condition into a denting point requirement. His method of proof follows Visintin's [31].

In this paper I will demonstrate that all four results mentioned above, including the one by Rzeżuchowski, follow from one central result, Theorem 1.4, to be stated below.

Let E be a Banach space, equipped with a norm $\|\cdot\|$. We shall also consider on E a given locally convex Hausdorff topology τ which is not stronger than the norm topology, and not weaker than the usual weak topology $\sigma(E, E')$. Here E' stands for the topological dual of E; the corresponding bilinear form will be denoted by $<\cdot,\cdot>$, as usual.

Let (u_k) be a sequence in the space $\mathcal{L}^1_E(\mu)$ of E-valued Bochner integrable functions on $(\Omega, \mathcal{F}, \mu)$. Recall that the dual of $\mathcal{L}^1_E(\mu)$ for the usual L^1-seminorm can be identified with the set $\mathcal{L}^\infty_{E'}[E](\mu)$ of all E'-valued scalarly measurable bounded functions on Ω (here both spaces are prequotient spaces) [18, VII.4]. The sequence (u_k) is said to converge *weakly* to u_0 in $\mathcal{L}^1_E(\mu)$ if

$$\lim_{k\to\infty}\int_\Omega < u_k(\omega) - u_0(\omega), b(\omega) > \mu(d\omega) = 0$$

for every $b \in \mathcal{L}^\infty_{E'}[E](\mu)$. Note that then (u_k) is relatively weakly compact, and this implies by [10, Thm. 1, Remark 1] that

$$\int_{(\cdot)} \|u_k\| d\mu \quad \text{is equi-absolutely continuous with respect to } \mu,$$

and that for every $\epsilon > 0$ there exists a set $F_\epsilon \in \mathcal{F}, \mu(F_\epsilon) < +\infty$, such that

$$\sup_k \int_{\Omega\setminus F_\epsilon} \|u_k\| d\mu \leq \epsilon.$$

Recall that the definition of uniform integrability can easily be extended from finite to arbitrary measure spaces in the following way. A sequence $(\phi_k) \subset \mathcal{L}^1_{\mathbf{R}}(\mu)$ is said to be **uniformly integrable** if for every $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{L}^1_{\mathbf{R}}(\mu)$ such that

$$\sup_k \int_{\{|\phi_k|\geq\phi_\epsilon\}} |\phi_k| d\mu \leq \epsilon.$$

In particular, it follows from the above that when $(u_k) \subset \mathcal{L}^1_E(\mu)$ converges weakly, then $(\|u_k\|)$ is uniformly integrable. For by [15, II.19] there exists, for arbitrary $\epsilon > 0$, a constant c_ϵ with

$$\sup_k \int_{F_{\epsilon/2} \cap \{\|u_k\| \geq c_\epsilon\}} \|u_k\| d\mu \leq \epsilon/2.$$

Then $\phi_\epsilon := c_\epsilon 1_{F_{\epsilon/2}}$ satisfies the definition of uniform integrability for $(\|u_k\|)$.

Next, recall from appendix A that the sequence consisting of the relaxations ϵ_{u_k} of the functions $u_k, k \in \mathbf{N}$, is said to be *tight* [2] – with respect to the given topology τ – if there exists a function $h : \Omega \times E \to [0, +\infty]$ such that

(i) for a.e. ω the function $h(\omega, \cdot)$ is inf-compact on E_τ (i.e., for every $\beta \in \mathbf{R}$ the set $\{x \in E : h(\omega, x) \leq \beta\}$ is τ-compact).

(ii) $\sup_k \int_\Omega^* h(\omega, u_k(\omega))\mu(d\omega) < +\infty$.

Here outer integration is used (cf. Appendix A).

Based on Appendix A, it is shown in section 2 that when $(u_k) \subset \mathcal{L}^1_E(\mu)$ converges weakly to u_0 and (ϵ_{u_k}) is tight for the topology τ,* then not only the classical limit property

$$u_0(\omega) \in \cap_{p=1}^\infty \text{ cl co}\{u_k(\omega) : k \geq p\} \quad \text{a.e.}$$

holds (by Mazur's theorem), but also the following, much stronger property

$$u_0(\omega) \in \text{ cl co Ls}_\tau(u_k(\omega)) \quad \text{a.e.}$$

which in case E is finite-dimensional can be strengthened further into

$$u_0(\omega) \in \text{ co Ls}(u_k(\omega)) \quad \text{a.e.}$$

Here $\text{Ls}_\tau(X_k)$ denotes the **limes superior** (in the sense of Kuratowski) of a sequence (X_k) of subsets of E, defined by

$$\text{Ls}_\tau(X_k) := \cap_{p=1}^\infty \text{cl}_\tau(\cup_{k=p}^\infty X_k).$$

When all sets X_k are singletons (say $X_k = \{x_k\}$), I shall also write $\text{Ls}_\tau(x_k)$ instead of $\text{Ls}_\tau(\{x_k\})$, as was done above.

The main result of this paper can now be stated:

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*Such tightness will hold automatically if E is reflexive and if $\tau = \sigma(E, E')$; cf. Remark 2.2.

THEOREM 1.4: *Let (u_k) be a sequence in $\mathcal{L}^1_E(\mu)$ such that*

$$u_k \to u_0 \quad \text{weakly in } \mathcal{L}^1_E(\mu).$$

Suppose that (ϵ_{u_k}) is tight with respect to the topology τ on E and that

$$u_0(\omega) \in \partial_e \text{cl co Ls}_\tau(u_k(\omega)) \quad \text{a.e.}$$

Then

$$\epsilon_{u_k} \Longrightarrow \epsilon_{u_0}.$$

Recall from Appendix A that by its definition the weak convergence statement $\epsilon_{u_k} \Longrightarrow \epsilon_{u_0}$ means the following: for every function $g : \Omega \times E \to [0, +\infty]$ such that $g(\omega, \cdot)$ is l.s.c. on E_τ for a.e. ω

$$\liminf_{k\to\infty} \int^*_\Omega g(\omega, u_k(\omega))\mu(d\omega) \geq \int^*_\Omega g(\omega, u_0(\omega))\mu(d\omega).$$

However, rather more can be said under the present circumstances:

COROLLARY 1.5: *Under the conditions of Theorem 1.4 the following are equivalent: i. The weak convergence statement*

$$\epsilon_{u_k} \Longrightarrow \epsilon_{u_0}.$$

ii. The inequality

$$\liminf_{k\to\infty} \int^*_\Omega g(\omega, u_k(\omega))\mu(d\omega) \geq \int^*_\Omega g(\omega, u_0(\omega))\mu(d\omega)$$

for every function $g : \Omega \times E \to (-\infty, +\infty]$ such that for a.e. ω

$$g(\omega, \cdot) \text{ is } \tau\text{-l.s.c. at } u_0(\omega) \text{ relative to } \{u_k(\omega) : k \geq 0\},$$

$$g(\omega, \cdot) \quad \text{is Borel measurable on } E_\tau,$$

with the following bound from below

$$g(\omega, x) \geq -C\|x\| + \psi(\omega)$$

for some $C \geq 0$ and $\psi \in \mathcal{L}^1_{\mathbf{R}}(\mu)$.

Proof: Of course, (ii) immediately implies (i). The converse follows by Proposition A.9. □

Consequently, for $E := \mathbf{R}^d$ Visintin's Theorem 1.2 follows directly from Theorem 1.4: simply use $g(\omega, x) := -|x - u_0(\omega)|$ in the above corollary, and note that (ϵ_{u_k}) is automatically tight, since $\sup_k \int h(\cdot, u_k) d\mu < +\infty$ for $h(\omega, x) := |x|$ (cf. Remark 2.2).

2. Proof of the Main Result

In this section Theorem 1.4 will be proven, using the Young measure theory developed in Appendix A.

Let me note beforehand that by the Pettis measurability theorem [16, II.1] there corresponds to each u_k a null set N_k such that $u_k(\Omega \backslash N_k)$ is a separable subset of E. Then, clearly, the closure of the linear span of all $u_k(\Omega \backslash N_k)$, $k \in \mathbf{N} \cup \{0\}$ is a separable Banach space, to which all considerations can be restricted. Of course, this means that I may suppose the Banach space E itself to be separable.

As a consequence, $E_{\|\cdot\|}$ is a Polish space, so E_τ is a Suslin locally convex space and $\mathcal{B}(E_{\|\cdot\|}) = \mathcal{B}(E_\tau) = \mathcal{B}(E_\sigma)$ by [27, Cor. 2 of Thm. II.10] (or by an easy ad hoc proof). Thus, one need not distinguish between $\mathcal{B}(E_\tau)$ and $\mathcal{B}(E_{\|\cdot\|})$, or between $\mathcal{P}(E_\tau)$ and $\mathcal{P}(E_{\|\cdot\|})$.

Recall that the *barycenter* bar $\nu \in E$ of a probability measure $\nu \in \mathcal{P}(E)$ is defined by

$$\text{bar } \nu := \int_E x \nu(dx),$$

provided that it exists, i.e., provided that $\int_E \|x\| \nu(dx) < +\infty$. The key to my proof of Theorem 1.4 is the following simple and intuitively appealing lemma [3] (under additional compactness conditions this result is well-known in Choquet theory).

LEMMA 2.1: *Suppose that for $\nu \in \mathcal{P}(E)$*

$$\textit{bar } \nu \in \partial_e(\textit{rm cl co supp}\nu).$$

Then ν is the Dirac measure concentrated at the point bar $\nu \in E$.

Proof: For every closed convex $D \subset E$ with bar $\nu \notin D$ one has $\nu(D) = 0$. For if it were true that $\nu(D) > 0$, then surely $\nu(D) < 1$ (or else the barycenter of ν would lie in D). Hence, $\nu = \nu(D)\nu_1 + (1 - \nu(D))\nu_2$, where $\nu_1, \nu_2 \in \mathcal{P}(E)$ are defined as the normalized restrictions of ν to D and $E \backslash D$ respectively. Thus, one would then find bar $\nu = \nu(D)$ bar $\nu_1 + (1 - \nu(D))$ bar ν_2. So by the extremality property of bar ν this would imply bar $\nu =$ bar ν_1. But this contradicts the assumption bar $\nu \notin D$, as bar ν_1 belongs to the closed convex set D. In particular, I conclude now that every closed ball in E not containing bar ν has measure zero under ν. So the same holds then also for open balls. Finally, then, it follows by second countability of $E_{\|\cdot\|}$ that the norm-open set $E\backslash\{\text{ bar } \nu\}$ has measure zero under ν. □

Proof of Theorem 1.4: Let (l) be an arbitrary subsequence of (k). By Theorem A.5 there exist a subsequence (m) of (l) and a Young measure $\delta_* \in \mathcal{R}_{E_\tau}(\mu)$ such that

$$\epsilon_{u_m} \xrightarrow{K} \delta_*.$$

By Corollary A.2 this gives

$$\text{supp}\, \delta_*(\omega) \subset \text{Ls}_\tau(u_m(\omega)) \subset \text{Ls}_\tau(u_k(\omega)) \quad \text{a.e.} \tag{2.1}$$

By Proposition A.8, applied to the function $g(\omega, x) := \|x\|$ (note that the norm functional is τ-l.s.c.), it follows that

$$\int_\Omega [\int_E \|x\|\delta_*(\omega)(dx)]\mu(d\omega) = I_g(\delta_*) \leq \sup_m I_g(\epsilon_{u_m}) < +\infty.$$

Consequently, bar $\delta_*(\omega)$ has to exist a.e., and of course now

$$\text{bar } \delta_*(\omega) \in \text{cl co Ls}_\tau(u_k(\omega)).$$

Since E_τ is Suslin locally convex, there exists a countable collection $\{x'_j\}$ in E' which separates the points of E [13, III.31]. I apply Lemma A.7 to $g'(\omega, x) := 1_B(\omega) < x, x'_j >$, for arbitrary $B \in \mathcal{F}$ and $j \in \mathbf{N}$. As $(\|u_k\|)$ is uniformly integrable (section 1), it is easy to see that both g' and $-g'$ satisfy the conditions of that lemma. Therefore,

$$I_{g'}(\delta_*) = \lim_{m\to\infty} I_{g'}(\epsilon_{u_m}).$$

By the weak convergence of (u_k) to u_0 this gives

$$\int_B [\int_E < x, x'_j > \delta_*(\omega)(dx)]\mu(d\omega) = \int_B < u_0(\omega), x'_j > \mu(d\omega),$$

so, B being arbitrary, I conclude that for every j

$$< \text{bar } \delta_*(\omega), x'_j >=< u_0(\omega), x'_j > \quad \text{a.e.}$$

Because (x'_j) separates the points of E, it follows that

$$\text{bar } \delta_*(\omega) = u_0(\omega) \quad \text{a.e.}$$

[Let me pause briefly to observe that by (2.1) this means that

$$u_0(\omega) \in \text{cl co Ls}_\tau(u_m(\omega)) \quad \text{a.e.}$$

and for finite-dimensional E by [24] even

$$u_0(\omega) \in \text{co Ls}_\tau(u_m(\omega)) \quad \text{a.e.}$$

as I already stated before.]

By Lemma 2.1, in view of the extremality hypothesis, the above implies

$$\delta_*(\omega) = \epsilon_{u_0}(\omega) \quad \text{a.e.}$$

It has now been established that every subsequence (l) of (k) contains a further subsequence (m) such that

$$\epsilon_{u_m} \xrightarrow{K} \epsilon_{u_0},$$

whence *a fortiori*

$$\epsilon_{u_m} \Longrightarrow \epsilon_{u_0}$$

by Proposition A.8. Thus, the sequence (ϵ_{u_k}) as a whole must converge weakly to ϵ_{u_0}. □

An interesting open question is whether the weak convergence result of Theorem 1.4 can be strengthened into a result on K-convergence:

$$\epsilon_{u_k} \xrightarrow{K} \epsilon_{u_0}.$$

Remark 2.2: In case E is reflexive and $\tau = \sigma(E, E')$, two improvements can be introduced in Theorem 1.4 and its corollary.

First, in this case tightness is an automatic consequence of the uniform integrability of $(\|u_k\|)$ (which, in turn, is a consequence of the weak convergence of (u_k) to u_0; cf. section 1). For obviously

$$\sup_k I_h(\epsilon_{u_k}) = \sup_k \int_\Omega \|u_k\| d\mu < +\infty$$

will hold for $h(\omega, x) := \|x\|$, which is then inf-compact in x.

Second, Corollary 1.5 can then be strenghtened as follows: Instead of

$$g(\omega, \cdot) \text{ is l.s.c. at } u_0(\omega) \text{ relative to } \{u_k(\omega) : k \geq 0\},$$

it is enough to require

$$g(\omega, \cdot) \text{ is sequentially l.s.c. at } u_0(\omega) \text{ relative to } \{u_k(\omega) : k \geq 0\}.$$

Indeed, since E is a Suslin locally convex space, E' is separable for $\sigma(E', E)$ by [13, III.32]. So by Šmulian's theorem [17, 3.2] it follows that sequential and ordinary compactness in E_σ are the same. Therefore, for every $\epsilon > 0$ the function $g_\epsilon := g + \epsilon h$ satisfies the conditions of Lemma A.7. An obvious limit argument, in which ϵ goes to zero, then leads to the desired result.

Remark 2.3: When τ is the norm-topology on E (as is the case for finite-dimensional E), it is interesting to observe that the convergence result $\epsilon_{u_k} \Longrightarrow \epsilon_{u_0}$ of Theorem 1.4 is **equivalent** to

$$(u_k) \text{ converges locally in measure to } u_0.$$

This is seen by applying the definition of weak convergence to the function

$$g(\omega, x) := \begin{cases} -1 & \text{if } \omega \in D \text{ and} \|x - u_0(\omega)\| \geq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

for arbitrary $D \in \mathcal{F}, \mu(D) < +\infty$, and $\epsilon > 0$. This gives the desired local convergence in measure:

$$\limsup_k \mu(\{\omega \in D : \|u_k(\omega) - u_0(\omega)\| \geq \epsilon\}) \leq 0.$$

Conversely, given such local convergence in measure, every subsequence of (u_k) has a subsequence converging a.e. to u_0, which makes the proof of lower semi-continuity for the desired integral functionals a simple application of Fatou's lemma.

Remark 2.4: Remark A.6 implies that Theorem 1.4 remains valid if one works with different, ω-dependent topologies τ_ω on E (which itself may have ω-dependent norms), provided that E is a separable Banach space, and for each ω the topology τ_ω on E is not weaker than the weak topology and not stronger than the norm topology.

Theorem 1.4 can immediately be generalized into a version for a sequence of *scalarly* integrable functions (u_k). Suppose that E is a Banach space with norm $\|\cdot\|$ and with a locally convex Suslin topology τ, not stronger than the norm topology and not weaker than the topology $\sigma(E, E')$. Then by [13, III.32] the dual E' of E_τ has a countable subset (x'_j) which is $\sigma(E', E)$-dense in the unit ball of E'. Let $\mathcal{L}^1_E(\mu)[E']$ be the set of all *scalarly integrable* functions $u : \Omega \to E$, i.e. such that

$$< u(\cdot), x' > \in \mathcal{L}^1_{\mathbf{R}}(\mu) \quad \text{for every } x' \in E'.$$

THEOREM 2.5: *Let (u_k) be a sequence in $\mathcal{L}^1_E(\mu)[E']$ such that for every $j \in \mathbf{N}, B \in \mathcal{F}$*

$$\lim_{k \to \infty} \int_B < u_k - u_0, x'_j > d\mu = 0.$$

Suppose that (u_k) is tight with respect to the topology τ on E and that

$$u_0(\omega) \in \partial_e \mathrm{cl\ co\ Ls}_\tau(u_k(\omega)) \quad \text{a.e.}$$

(2.2) *$Ls(u_k(\omega))$ is either norm-separable or τ-compact a.e.*

Then

$$\epsilon_{u_k} \Longrightarrow \epsilon_{u_0}.$$

Proof: The proof is almost the same as that of Theorem 1.4. The only real difference lies in the barycentric argument involving Lemma 2.1. Here existence of the barycenters bar $\delta_*(\omega)$ can only be guaranteed thanks to the provision (2.2) (this is the sole **raison d'être** for that condition), whereas in the proof of Theorem 1.4 one could assume without loss of generality that E is separable.

COROLLARY 2.6: *Under the conditions of Theorem 2.5 and the additional condition*

$$(\|u_k\|) \text{ is uniformly integrable}$$

the following are equivalent:

i. The weak convergence statement

$$\epsilon_{u_k} \Longrightarrow \epsilon_{u_0}$$

ii. The inequality

$$\liminf_{k\to\infty} \int_\Omega^* g(\omega, u_k(\omega))\mu(d\omega) \geq \int_\Omega^* g(\omega, u_0(\omega))\mu(d\omega)$$

for every function $g : \Omega \times E \to (-\infty, +\infty]$ such that for a.e. ω

$$g(\omega, \cdot) \text{ is } \tau\text{-l.s.c. at } u_0(\omega) \text{ relative to } \{u_k(\omega) : k \geq 0\},$$

$$g(\omega, \cdot) \text{ is Borel measurable on } E_\tau,$$

with the following bound from below

$$g(\omega, x) \geq -C\|x\| + \psi(\omega)$$

for some $C \geq 0$ and $\psi \in \mathcal{L}^1_{\mathbf{R}}(\mu)$.

The proof of this corollary is quite analogous to that of Corollary 1.5, and will be omitted.

3. Applications

In this section I shall apply Theorems 1.4 and 2.5 to problems considered by myself [3], Castaing [12] and Rzeżuchowski [26].

My first application is a refinement of the infinite-dimensional extension in [3, Thm.1] of Visintin's result.

PROPOSITION 3.1: *Suppose that the Banach space E is reflexive, and let (u_k) be a sequence in $\mathcal{L}^1_E(\mu)$ such that*

$$u_k \to u_0 \quad \textit{weakly in } \mathcal{L}^1_E(\mu).$$

Suppose that

$$u_0(\omega) \in \partial_e \mathrm{cl}\ \mathrm{co}\ \mathrm{Ls}_\sigma(u_k(\omega)) \quad \text{a.e.}$$

Then

$$\lim_{k\to\infty} \int_\Omega g(\omega, u_k(\omega) - u_0(\omega))\mu(d\omega) = 0$$

for every $\mathcal{F} \times \mathcal{B}(E_\tau)$-measurable function $g : \Omega \times E \to \mathbf{R}$ such that for a.e. ω

$$g(\omega, \cdot) \textit{ is sequentially } \sigma(E, E')\textit{-continuous at } u_0(\omega) \textit{relative to} \{u_k(\omega) : k \geq 0\},$$

$$g(\omega, 0) = 0,$$

with the following bound:

$$|g(\omega, x)| \leq C\|x\| + \psi(\omega)$$

for some $C \geq 0$ and $\psi \in \mathcal{L}^1_{\mathbf{R}}(\mu)$.

Proof: Take $\tau := \sigma(E, E')$. In view of Remark 2.2 the result follows directly from applying Corollary 1.5 to both g and $-g$. □

In comparison to [3], the following improvements have been made: (i) the extreme point condition is slightly relaxed (this possibility, which is implicit in the proof of [3], was already signaled by Valadier in [29, thm.21]), (ii) the continuity condition has been localised.

Next, I show how Theorem 1.4 implies a very recent result by Rzeżuchowski [26]. (His method of proof follows Visintin's original proof in [31].)

Recall that for $D \subset E$ an element $x_0 \in D$ is said to be a *denting point* for D [16, p. 270] if

$$x_0 \notin \text{cl co } (D \backslash B(x_0; \epsilon)) \quad \text{for all } \epsilon > 0.$$

Here $B(x_0; \epsilon) := \{x \in E : \|x - x_0\| < \epsilon\}$. The collection of all denting points of $D \subset E$ is denoted by $\partial_d D$; it is simple to show that always

$$\partial_d D \subset \partial_e D.$$

PROPOSITION 3.2: *Suppose that the Banach space E is reflexive. Let (u_k) be a sequence in $\mathcal{L}_E^1(\mu)$ such that*

$$u_k \to u_0 \quad \text{weakly in } \mathcal{L}_E^1(\mu).$$

Suppose that

$$u_0(\omega) \in \partial_d \text{cl co } \{u_k(\omega) : k \geq 0\} \quad \text{a.e.}$$

Then strong convergence in $\mathcal{L}_E^1(\mu)$ holds:

$$\lim_{k \to \infty} \int_\Omega \|u_k - u_0\| : d\mu = 0.$$

Only the first part of the following characterization of denting points will be used in the proof of the above result (the second part serves as a completion). See also [14, Prop. 25.13], [23,30] .

LEMMA 3.3: (a) *Let $D \subset E, x_0 \in D$. Then $x_0 \in \partial_d(D)$ implies that the identity mapping $\iota : E_\sigma \to E_{\|\cdot\|}$ is continuous at x_0, relative to D.*
(b) *Let $D \subset E$ be closed convex, $x_0 \in \partial_e(D)$. Suppose that $D \cap \text{cl} B(x_0; \epsilon_0)$ is weakly compact for some $\epsilon_0 > 0$. Then the converse of the implication in a is also true.*

Proof: (a) Let $\epsilon > 0$ be arbitrary. By dentability, it follows from the Hahn-Banach theorem that there exists an open half-space $H \subset E$ such that $H \ni x_0$ and $E \backslash H \supset D \backslash B(x_0; \epsilon)$. Evidently, this means that $\|x - x_0\| < \epsilon$ for all x in the relative weak neighborhood $H \cap D$ of x_0.
(b) Let $0 < \epsilon < \epsilon_0$ be arbitrary. Define $D_0 := D \cap \text{cl } B(x_0; \epsilon)$; then D_0 is weakly compact; also, $x_0 \in \partial_e D_0$. By [14, 25.13], x_0 has, relative to D_0, a weak neighborhood basis consisting of open half spaces. By the continuity property of the mapping ι, the point x_0 has, relative to D, a neighborhood basis

which consists of weakly open sets. Hence, there exists an open half space H_0, containing x_0, such that $H_0 \cap D_0 \subset B(x_0;\epsilon) \cap D$. The result is now a direct consequence of the following observation: if $x \in H_0 \cap D$ has $\|x - x_0\| > \epsilon$, then $y := x_0 + \epsilon(x - x_0)/\|x - x_0\|$ belongs to $H_0 \cap (D_0 \backslash B(x_0, \epsilon))$ (since such y cannot exist, it follows that $D \backslash B(x_0, \epsilon)$ is contained in $E \backslash H_0$). □

Proof of Proposition 3.2: . Corollary 1.5 can be applied, with $\tau = \sigma(E, E')$, for by Remark 2.2 the tightness condition holds and the extreme point condition is fulfilled, thanks to the denting point condition for the values of u_0. By this corollary, applied to the function $g(\omega, x) := -\|x - u_0(\omega)\|$, the result follows. (Note that by Lemma 3.3a g satisfies the local lower semicontinuity of Theorem 1.4). □

In Proposition 3.2 it is not enough to have the denting point condition

$$u_0(\omega) \in \partial_d \mathrm{cl\ co\ Ls}_\sigma(u_k(\omega)) \quad \text{a.e.},$$

as was pointed out to me by M. Valadier in response to an earlier, erroneous version of that result. For instance, consider ℓ^2 with the basis of unit vectors (e_k). Then for constant functions $u_{2k} \equiv 0$ and $u_{2k+1} \equiv e_k$ one has weak convergence to 0 – assuming the measure space is finite – and even $\mathrm{Ls}_{\|\cdot\|}(u_k(\omega)) = \{0\}$ a.e.; yet strong convergence does not hold. This example comes from [1].

Next, I state a new result for a separable Banach space E, equipped with a topology τ which is not weaker than $\sigma(E, E')$ and not stronger than the norm-topology. It includes also a result of Castaing [12, Thm.2.1]. Before stating it, I recall that $D \subset E$ is said to be *τ-compact with respect to closed balls* if $D \cap \mathrm{cl}\ B(0;\beta)$ is τ-compact for every $\beta \geq 0$.

PROPOSITION 3.4: *Let (u_k) be a sequence in $\mathcal{L}^1_E(\mu)$ such that*

$$u_k \to u_0 \quad \textit{weakly in } \mathcal{L}^1_E(\mu),$$

Suppose that

$$u_0(\omega) \in \partial_e \mathrm{cl\ co\ Ls}_\tau(u_k(\omega)) \quad \text{a.e.}$$

and

$$\mathrm{cl}_\tau\{u_k(\omega) : k \in \mathbf{N}\} \textit{ is } \tau\textit{-compact with respect to closed balls a.e.}$$

Then

$$\lim_{k\to\infty} \int_\Omega g(\omega, u_k(\omega) - u_0(\omega))\mu(d\omega) = 0$$

for every $\mathcal{F} \times \mathcal{B}(E_\tau)$-measurable function $g : \Omega \times E \to \mathbf{R}$ such that for a.e. ω

$$g(\omega) \text{ is } \tau\text{-continuous at } u_0(\omega) \text{ relative to } \{u_k(\omega) : k \geq 0\},$$

$$g(\omega, 0) = 0,$$

with the following bound:

$$|g(\omega, x)| \leq C\|x\| + \psi(\omega)$$

for some $C \geq 0$ and $\psi \in \mathcal{L}^1_{\mathbf{R}}$.

Proof: Apply Theorem 1.4. Define

$$h(\omega, x) = \begin{cases} \|x\| & \text{if } x \in \text{cl}\{u_k(\omega) : k \in \mathbf{N}\}, \\ +\infty & \text{otherwise} \end{cases}$$

Then for every ω the function $h(\omega, \cdot)$ is inf-compact (note that for every $\beta \in \mathbf{R}$ the set $\{x \in E : h(\omega, x) \leq \beta\}$ is compact, being the intersection of the set $\text{cl}\{u_k(\omega) : k \in \mathbf{N}\}$ and $\text{cl } B(0; \beta)$). Hence, the tightness condition of Theorem 1.4 is fulfilled because of

$$\sup_{k \in \mathbf{N}} I_h(\epsilon_{u_k}) = \sup_k \int_\Omega \|u_k\| d\mu < +\infty.$$

The remaining details, which go as in the proof of Proposition 3.1, are left to the reader. □

Note that in [12, Thm.2.1] compactness is required instead of of compactness with respect to closed balls, and E is supposed to be separable. Moreover, some measurability and convexity conditions used in [12] have been removed.

I now discuss an application to the scalarly integrable case, which captures [12, Thm.2.2]. Here E is the dual of a separable Banach space F; the norm on E is the dual norm with respect to F. Let us equip E with the topology τ of uniform convergence on compacta (since the closed unit ball U of E is metrizable and τ-compact, E_τ is a locally convex Suslin space). Note that on U the topology τ equals the weak star topology $\sigma(E, F)$.

PROPOSITION 3.5: *Let (u_k) be a sequence in $\mathcal{L}^1_E(\mu)[F]$ such that*

$$u_k \to u_0 \quad \textit{weakly in } \mathcal{L}^1_E(\mu)[F].$$

Suppose that

$$u_0(\omega) \in \partial_e \text{cl co Ls}_{\sigma(E,F)}(u_k(\omega)) \quad \text{a.e.}$$

and that for some scalar function $g \in \mathcal{L}^1_{\mathbf{R}}(\mu)$

$$\{u_k(\omega) : k \in \mathbf{N}\} \subset g(\omega)U \quad \text{a.e.}$$

Then

$$\epsilon_{u_k} \Longrightarrow \epsilon_{u_0}.$$

Proof: Apply Theorem 2.5 with $\tau := \sigma(E,F)$. By the Alaoglu–Bourbaki theorem the set cl co $\{u_k(\omega) : k \geq 0\}$ is τ-compact for a.e. ω. Hence, (ϵ_{u_k}) is tight, as is seen by using h given by

$$h(\omega, x) = \begin{cases} 0 & \text{if } x \in \text{cl co } \{u_k(\omega) : k \in \mathbf{N}\}, \\ +\infty & \text{otherwise.} \end{cases}$$

The result then follows immediately. □

Finally, I give the following regularity result (in the sense of well-posedness à la Tychonov). This result, stated for a Banach space E, generalizes [31, Thm.8].

THEOREM 3.6: *Let* $f : \Omega \times E \to [0, +\infty]$ *be* $\mathcal{F} \times \mathcal{B}(E)$*-measurable and such that for a.e.* ω *the function* $f(\omega, \cdot)$ *is* $\sigma(E, E')$*-inf-compact and strictly convex. Also, let* $b \in \mathcal{L}^\infty_{E'}[E](\mu)$ *be given, and consider the minimization problem* (P)*:*

$$\min_{u \in \mathcal{L}^1_E(\mu)} J(u),$$

where

$$J(u) := \int_\Omega f(\omega, u(\omega))\mu(d\omega) - \int_\Omega < u(\omega), b(\omega) > \mu(d\omega).$$

Suppose that f *has the following superlinear growth property: for every* $\epsilon > 0$ *there exists* $\psi_\epsilon \in \mathcal{L}^1_{\mathbf{R}}(\mu)$, $\psi_\epsilon \geq 0$, *such that for a.e.* ω

$$\|x\| \geq \psi_\epsilon(\omega) \quad \text{implies } \epsilon f(\omega, x) \geq \|x\|.$$

Then the infimum value ι *of* (P) *is attained by a unique* $u_* \in \mathcal{L}^1_E(\mu)$. *Moreover, if* $\iota < +\infty$ *then for any minimizing sequence* (u_k) *for* (P) *the following strong convergence results hold:*

$$\int_\Omega \|u_k(\omega) - u_*(\omega)\|\mu(d\omega) \to 0,$$

$$\int_\Omega |f(\omega, u_k(\omega)) - f(\omega, u_*(\omega)|\mu(d\omega) \to 0.$$

Proof: If $\iota = +\infty$, every $u \in \mathcal{L}^1_E(\mu)$ is optimal. So suppose from now on that $\iota < +\infty$. Let (u_k) be any minimising sequence, and let (l) be any subsequence of (k) (so (u_l) is also minimising). Then by [6, Thm.4.1] (l) contains at least one subsequence (m) such that (u_m) converges weakly to some u_* in $\mathcal{L}^1_E(\mu)$.* Moreover, since J is evidently strictly convex and strongly l.s.c. on $\mathcal{L}^1_E(\mu)$, it must also be weakly l.s.c. This shows that $\iota = \liminf_m J(u_m) \geq J(u_*)$. Note that the minimising u_* must be unique (by strict convexity).

But one gets more than that: the same reasoning leads to

$$\liminf_m \int_B f(\omega, u_m(\omega))\mu(d\omega) \geq \int_B f(\omega, u_*(\omega))\mu(d\omega)$$

for every $B \in \mathcal{F}$. By the obvious convergence

$$\lim_m \int_\Omega f(\cdot, u_m(\cdot))d\mu = \iota - \int_\Omega < b, u_* > d\mu = \int_\Omega f(\cdot, u_*(\cdot))d\mu$$

it also follows that $(f(\cdot, u_m(\cdot)))$ converges weakly in $\mathcal{L}^1_{\mathbf{R}}(\mu)$ to $f(\cdot, u_*(\cdot))$. So now

$$(u_m, f(\cdot, u_m(\cdot))) \to (u_*, f(\cdot, u_*(\cdot)) \quad \text{weakly in } \mathcal{L}^1_{E\times\mathbf{R}}(\mu).$$

For a.e. ω it is evident, by strict convexity of $f(\omega, \cdot)$, that $(u_*(\omega), f(\omega, u_*(\omega)))$ is an extreme point of the epigraph of $f(\omega, \cdot)$, which itself is closed and convex. Thus, one certainly has

$$(u_*(\omega), f(\omega, u_*(\omega)) \in \partial_e \,\mathrm{cl}\,\mathrm{co}\,\mathrm{Ls}_\sigma((u_m(\cdot), f(\omega, u_m(\omega))) \quad \text{a.e.}$$

Using Theorem 1.4, I now conclude that every subsequence (l) of (k) has a further subsequence (m) for which

$$(u_m, f(\cdot, u_m(\cdot))) \to (u_*, f(\cdot, u_*(\cdot)) \quad \text{strongly in} \mathcal{L}^1_{E\times\mathbf{R}},$$

and this implies the desired strong convergence result of $(u_k, f(\cdot, u_k(\cdot)))$ to $(u_*, f(\cdot, u_*(\cdot)))$. □

*This generalizes a result known as Diestel's theorem; the proof in [6] is again based on K-convergence – but of another type.

Appendix A. A New Introduction to Young Measure Theory

In this appendix I present basic Young measure theory, in particular the weak convergence topology and the notion of tightness, from a new perspective: I derive these results from the classical theory for weak convergence of measures by means of K-**convergence**, a general, unifying notion of convergence for scalarly measurable functions introduced in [6,5]. Young measures are scalarly measurable functions, taking as their values ordinary probability measures. Since K-convergence concerns the **pointwise** convergence of arithmetic averages of Young measures, this has an obvious advantage for the reader: only some basic familiarity with weak convergence of probability measures [11,15] is expected, instead of knowledge of siseable parts from functional analysis and measure theory.

The framework of this appendix consists of the σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ encountered in the main text, and of a completely regular Suslin space S (recall that a Suslin space is the continous image of some Polish space [13,15,27]. The set of all probability measures on $(S, \mathcal{B}(S))$, is denoted by $\mathcal{P}(S)$; here $\mathcal{B}(S)$ stands for the Borel σ-algebra on S. Recall that the *support* $\operatorname{supp}\nu$ of $\nu \in \mathcal{P}(S)$ is defined as the intersection of all closed sets $F \subset S$ with $\nu(F) = 1$. Recall also that a sequence (or generalised sequence) (ν_k) in $\mathcal{P}(S)$ is said to converge weakly (or *narrowly*) to the probability measure ν_0 (notation: $\nu_k \Rightarrow \nu_0$) if

$$\lim_{k\to\infty} \int_S c\, d\nu_k = \int_S c\, d\nu_0 \quad \text{for every } c \in \mathcal{C}_b(S).$$

Here $\mathcal{C}_b(S)$ stands for the set of all bounded continuous real-valued functions on S. The following result is a consequence of the definition of the weak convergence topology.

PROPOSITION A.1: *Suppose that $\nu_k \Rightarrow \nu_0$ in $\mathcal{P}(S)$. Then*

$$\liminf_{k\to\infty} \int_S q\, d\nu_k \geq \int_S q\, d\nu_0.$$

for every measurable function $q : S \to (-\infty, +\infty]$ such that

$$q \text{ is l.s.c. at every point of } \operatorname{supp}\nu_0$$

relative to $\cup_{k=0}^{\infty}\operatorname{supp}\nu_k$, and

$$q \text{ is bounded from below by a constant.}$$

Proof: Define $S_0 := \cup_{k=0}^{\infty} \operatorname{supp}\nu_k$. Observe that S_0 is a completely regular Hausdorff space for the relative topology. Let $\bar{q} : S_0 \to (-\infty, +\infty]$ be the l.s.c. hull of q relative to S_0, i.e. the largest l.s.c. function on S_0 nowhere larger than q. Then by the fact that $S_0 \supset \operatorname{supp}\nu_k, k \in \mathbf{N} \cup \{0\}$ and by [15, III.55]

$$\liminf_k \int_S q\, d\nu_k \geq \liminf_k \int_{S_0} \bar{q}\, d\nu_k \geq \int_{S_0} \bar{q}\, d\nu_0.$$

By hypothesis, $\nu_0(\{s \in S_0 : q(s) = \bar{q}(s)\}) = 1$. This leads to

$$\int_{S_0} \bar{q}\, d\nu_0 = \int_{S_0} q\, d\nu_0 = \int_S q\, d\nu_0,$$

since $S_0 \supset \operatorname{supp}\nu_0$. Thus, the inequality has been proven. □

COROLLARY A.2: *Suppose that $\nu_k \Rightarrow \nu_0$ in $\mathcal{P}(S)$. Then*

$$\operatorname{supp}\nu_0 \subset \operatorname{Ls}(\operatorname{supp}\nu_k).$$

Moreover, if for some sequence $(\pi_k) \subset \mathcal{P}(S)$

$$\frac{1}{n}\sum_{k=1}^{n} \pi_k \Rightarrow \nu_0,$$

then

$$\operatorname{supp}\nu_0 \subset \operatorname{Ls}(\operatorname{supp}\pi_k).$$

Proof: Define $S_p := \cup_{k=p}^{\infty} \operatorname{supp}\nu_k$. Apply Proposition A.1 to the l.s.c. function $q_p : S \to \{0, +\infty\}, p \in \mathbf{N}$, given by

$$q_p(s) := \begin{cases} 0 & \text{if } s \in \operatorname{cl} S_p, \\ +\infty & \text{otherwise.} \end{cases}$$

This gives $\int_S q_p : d\nu_0 = 0$. Therefore, $\nu_0(\operatorname{cl} S_p) = 1$, which amounts to $\operatorname{supp}\nu_0 \subset \operatorname{cl} S_p$.

Secondly, for (π_k) as stated one has $\sum_{k=p}^{n} \pi_k/(n-p+1) \Rightarrow \nu_0$ for every fixed $p \in \mathbf{N}$; hence, just as proven above,

$$\operatorname{supp}\nu_0 \subset \operatorname{cl} \cup_{k=p}^{\infty} \operatorname{supp}\pi_k,$$

since $\operatorname{supp}(\sum_{k=p}^{n} \pi_k/(n-p+1)) \subset \operatorname{cl} \cup_{k=p}^{n} \operatorname{supp}\pi_k$. The desired result then follows directly. □

Let me make the following observation: Because the space S is completely regular, the functions in $\mathcal{C}_b(S)$ separate the points of $\mathcal{P}(S)$ [15, III.54]. Since $\mathcal{P}(S)$ is also Suslin [27, Appendix, Thm.7], there exists a countable subset (c_j) of $\mathcal{C}_b(S)$ whose functions still separate the points of $\mathcal{P}(S)$ [13, III.31]. From the same fact it follows by [15, III.66] that every weakly compact subset of $\mathcal{P}(S)$ is metrizable, whence sequentially weakly compact.

In the terminology established by LeCam [21], a sequence (ν_k) in $\mathcal{P}(S)$ is said to be *tight* if for every $\epsilon > 0$ there exists a compact subset K_ϵ of S such that

$$\sup_k \nu_k(S \backslash K_\epsilon) \leq \epsilon,$$

or, equivalently, if there exists a function $h : S \to [0, +\infty]$ such that

(i) h is inf-compact on S,

(ii) $\sup_k \int_S h \, d\nu_k < +\infty$.

The following result can be found in [11, Appendix III, Thm.6] (it also follows by [15, III.55], taking into consideration the above remark on sequential weak compactness).

THEOREM A.3 (PROHOROV): *Suppose that the sequence (ν_k) in $\mathcal{P}(S)$ is tight. Then there exist a subsequence (m) of (k) and a probability measure $\nu_* \in \mathcal{P}(S)$ such that $\nu_m \Rightarrow \nu_*$.*

A *Young measure* with respect to Ω and S is defined to be a transition probability with respect to $(\Omega, \mathcal{F})$ and $(S, \mathcal{B}(S))$ [22, III.2]; that is to say, a Young measure is a function $\delta : \Omega \to \mathcal{P}(S)$ which is measurable with respect to the σ-algebra $\mathcal{F}$ on Ω and the Borel σ-algebra on $\mathcal{P}(S)$ corresponding to the weak convergence topology (the equivalence of this with the usual definition as in [22] follows by an easy monotone class argument; cf. [13, p. 103]. The set of all Young measures with respect to Ω and S is denoted by $\mathcal{R}_S(\mu)$. Particularly important are those Young measures which are associated to measurable functions $u : \Omega \to S$. In such a case the corresponding Young measure is denoted by ϵ_u; it is given by

$$\epsilon_u(\omega) := \text{ Dirac measure at } u(\omega).$$

The Young measure $\epsilon_u : \Omega \to \mathcal{P}(S)$ is called the **relaxation** of the measurable function $u : \Omega \to S$ (the seminal idea of completing the class of ordinary measurable functions in this way is due to Young [32]).

In this paper weak convergence of Young measures will come about as a byproduct of a new, stronger notion, called K-*convergence*. The origins of this notion lie in the following deep result by Komlós [20], which is valid for an arbitrary measure space $(\Omega, \mathcal{F}, \mu)$.

THEOREM A.4 (KOMLÓS): *Suppose that (ϕ_k) is a sequence in $\mathcal{L}^1_{\mathbf{R}}(\mu)$ such that*

$$\sup_k \int_\Omega |\phi_k| \, d\mu < +\infty.$$

Then there exist a subsequence (m) of (k) and an integrable function $\phi_* \in \mathcal{L}^1_{\mathbf{R}}(\mu)$ such that for every subsequence (m_i) of (m)

$$\frac{1}{n}\sum_{i=1}^{n} \phi_{m_i}(\omega) \to \phi_*(\omega) \quad \text{a.e.}$$

(the exceptional null set may depend upon the subsequence considered).

Following [6], a sequence (δ_k) in $\mathcal{R}_S(\mu)$ is said to **K-converge** to the Young measure δ_0 (notation: $\delta_k \xrightarrow{K} \delta_0$) if for every subsequence (k_i) of (k)

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{k_i}(\omega) \Rightarrow \delta_0(\omega) \quad \text{a.e.}$$

Note already that this entails

$$\operatorname{supp} \delta_0(\omega) \subset \text{ Ls } (\operatorname{supp}\delta_k(\omega)) \quad \text{a.e.}$$

by Corollary A.2. Other elementary consequences of K-convergence, connected with weak convergence in $\mathcal{R}_S(\mu)$, will be given shortly.

Following [6] a sequence (δ_k) in $\mathcal{R}_S(\mu)$ is said to be **tight** (alias **B-tight** [29]) if there exists a function $h : \Omega \times S \to [0, +\infty]$ such that

(i) $h(\omega, \cdot)$ is inf-compact on S for a.e. ω,

(ii) $\sup_k I^*_h(\delta_k) < +\infty$.

Here the following shorthand notation is used:

$$I^*_h(\delta) := \int^*_\Omega [\int_S h(\omega, s)\, \delta(\omega)(ds)] \mu(d\omega),$$

where $\int^*_\Omega$ denotes **outer integration**, which is recalled next:

For any – possibly nonmeasurable – function $\psi : \Omega \to (-\infty, +\infty]$ the **outer integral** of ψ over $(\Omega, \mathcal{F}, \mu)$ is defined by:

$$\int^*_\Omega \psi \, d\mu := \inf\{\int_\Omega \phi \, d\mu : \phi \in \mathcal{L}^1_{\mathbf{R}}(\mu), \phi \geq \psi \text{ a.e.}\}.$$

Here the infimum over the empty set equals $+\infty$ by definition.

Therefore, part (ii) of the above definition amounts precisely to the following: there exists a sequence $(\phi_{0,k})$ in $\mathcal{L}^1_{\mathbf{R}}(\mu)$ such that for every $k \in \mathbf{N}$:

$$0 \le \int_S h(\omega, s)\, \delta_k(\omega)(ds) \le \phi_{0,k}(\omega) \quad \text{a.e.} \tag{A.1}$$

and

$$\sup_k \int_\Omega \phi_{0,k}\, d\mu < +\infty. \tag{A.2}$$

Let me note as an aside that an equivalent definition of this form of tightness can also be given. Namely, by [19, Thm.2.4] (δ_k) is tight if and only if for every $\epsilon > 0$ there exists a compact-valued multifunction $\Gamma_\epsilon : \Omega \to 2^S$ such that

$$\sup_k \int_\Omega^* \delta_k(\omega)(S \backslash \Gamma_\epsilon(\omega))\mu(d\omega) \le \epsilon.$$

The following extension of Prohorov's Theorem A.3 to a criterion for relative compactness for K-convergence in $\mathcal{R}_S(\mu)$ was obtained in [6, Thm.5.1]. It forms a most important tool in this paper.

THEOREM A.5: *Suppose that the sequence (δ_k) in $\mathcal{R}_S(\mu)$ is tight. Then there exist a subsequence (m) of (k) and a Young measure $\delta_* \in \mathcal{R}_S(\mu)$ such that $\delta_m \xrightarrow{K} \delta_*$.*

Proof: Let $(c_j) \subset \mathcal{C}_b(S)$ be the separating set for $\mathcal{P}(S)$ defined before. By the fact that $(\Omega, \mathcal{F}, \mu)$ is σ-finite, there exist strictly positive functions in $\mathcal{L}^1_{\mathbf{R}}(\mu)$. Let $\bar{\phi}$ be one such function. For $j, k \in \mathbf{N}$ I define

$$\phi_{j,k}(\omega) := \bar{\phi}(\omega) \int_S c_j(s)\, \delta_k(\omega)(ds),$$

and I take $(\phi_{0,k})$ as in (A.1)-(A.2). Then it is clear that for every $j \in \mathbf{N} \cup \{0\}$

$$\sup_k \int_\Omega |\phi_{j,k}|\, d\mu < +\infty.$$

This makes it possible to apply Theorem A.4 repeatedly in a diagonal procedure (observe here the importance of the subsequence character of Theorem A.4). This yields a subsequence (m) of (k) and a sequence $(\phi_{j,*}) \subset \mathcal{L}^1_{\mathbf{R}}(\mu)$ such that for every $j \in \mathbf{N} \cup \{0\}$ and every subsequence (m_i) of (m)

$$\frac{1}{n} \sum_{i=1}^n \phi_{j,m_i}(\omega) \to \phi_{j,*}(\omega) \quad \text{a.e.}$$

Now this entails that for every subsequence (m_i) for a.e. ω:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \int_S h(\omega, s)\, \delta_{m_i}(\omega)(ds) \le \phi_{0,*}(\omega) < +\infty, \tag{A.3}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \int_S c_j(s)\, \delta_{m_i}(\omega)(ds) = \phi_{j,*}(\omega)/\bar{\phi}(\omega) \tag{A.4}$$

for every $j \in \mathbf{N}$ (here (A.3) follows by (A.1), and the identity results from dividing by $\bar{\phi}(\omega)$ and the above). Let us see first what this gives if (m) itself is considered as the subsequence in question. Fix $\omega \in \Omega$ outside the exceptional null-set. Define $(\nu_n) \subset \mathcal{P}(S)$ as follows:

$$\nu_n := \frac{1}{n}\sum_{m=1}^{n} \delta_m(\omega).$$

Then (ν_n) is tight in $\mathcal{P}(S)$, according to (A.3). By Prohorov's Theorem A.3 there exist at least one weakly convergent subsequence of (ν_m), and a corresponding limit point ν_*. Then by (A.4)

$$\int_S c_j\, d\nu_* = \phi_{j,*}(\omega) \quad \text{for every } j \in \mathbf{N}. \tag{A.5}$$

But by (A.4) for any other weakly convergent subsequence of (ν_n) the corresponding limit point will also have to satisfy (A.5). Since this equation uniquely determines the probability measure ν_*, it follows easily from (A.3) by Prohorov's theorem that the *whole* sequence (ν_n) converges weakly to $\delta_*(\omega) := \nu_* \in \mathcal{P}(S)$. By taking for $\delta_*(\omega)$ a fixed probability measure if ω belongs to the exceptional null set in (A.3)-(A.4), I obtain the desired Young measure δ_*. This whole argument can be repeated *verbatim* if I start out with an arbitrary subsequence (m_i) of (m). Note that, except for a possible shift of the exceptional null set (for which the statement of the theorem allows), the crucial relation (A.5) still will hold, regardless of the choice of subsequence. □

Remark A.6: From the pointwise nature of the above proof it is clear that in Theorem A.5 one can even allow for varying, ω-dependent topologies on S (and hence ω-dependent weak topologies on $\mathcal{P}(S)$). Of course, the sequence (c_j) can then also vary with ω; cf. [6].

For any function $g : \Omega \times S \to (-\infty, +\infty]$, measurable in its second variable and bounded from below by some function in $\mathcal{L}^1_{\mathbf{R}}(\mu)$, the **outer integral functional** $I_g^* : \mathcal{R}_S(\mu) \to (-\infty, +\infty]$ is defined by:

$$I_g^*(\delta) := \int_\Omega^* [\int_S g(\omega, s)\, \delta(\omega)(ds)]\mu(d\omega),$$

and in case outer integration can be replaced by ordinary integration I shall simply write $I_g(\delta)$.

An easy property of K-convergence is contained in the following Fatou-like lemma:

LEMMA A.7 (FATOU'S LEMMA FOR K-CONVERGENCE): *Suppose that* $\delta_k \xrightarrow{K} \delta_0$ *in* $\mathcal{R}_S(\mu)$. *Then*

$$\liminf_{k\to\infty} I_g^*(\delta_k) \geq I_g^*(\delta_0)$$

for every function $g : \Omega \times S \to (-\infty, +\infty]$ *such that for* a.e. ω,

$$g(\omega, \cdot) : S \to (-\infty, +\infty] \quad \text{is l.s.c. at every point of} \operatorname{supp} \delta_0(\omega)$$

relative to $\cup_{k=0}^{\infty} supp(\delta_k(\omega))$,

$$g(\omega, \cdot) : S \to (-\infty, +\infty] \quad \text{is measurable on } S,$$

and such that

$$g(\omega, s) \geq \phi_k(\omega) \quad \text{for all } s \in \operatorname{supp} \delta_k(\omega)$$

for some uniformly integrable sequence $(\phi_k) \subset \mathcal{L}^1_{\mathbf{R}}(\mu)$.

Proof: There exists a subsequence (l) of (k) such that the limes inferior, say α, equals $\lim_l I_g^*(\delta_l)$. Application of Komlós' Theorem A.4 to the sequence (ϕ_l) gives that there exist a subsequence (m) of (l) and $\phi_* \in \mathcal{L}^1_{\mathbf{R}}$ such that

$$\text{(A.6)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \phi_m(\omega) = \phi_*(\omega) \quad \text{a.e.}$$

Hence, by uniform integrability, it follows from the dominated convergence theorem that

$$\frac{1}{n} \sum_{m=1}^{n} \int_\Omega \phi_m d\mu \to \int_\Omega \phi_* d\mu,$$

so it follows that

$$\alpha - \int_\Omega \phi_* d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \int_\Omega^* (g_m - \phi_m) d\mu.$$

By Proposition A.1

$$\liminf_{m\to\infty} g_m(\omega) \geq g_0(\omega), \qquad \text{a.e.}$$

so by (A.6),

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} (g_m(\omega) - \phi_m(\omega)) \geq g_0(\omega) - \phi_*(\omega) \qquad \text{a.e.}$$

Here $g_m(\omega) := \int_S g(\omega, s)\delta_m(\omega)(ds) \geq \phi_m(\omega)$. Therefore, by Fatou's lemma

$$\alpha - \int_\Omega \phi_* d\mu \geq \int_\Omega^* (g_0 - \phi_*) d\mu = \int_\Omega^* g_0 d\mu - \int_\Omega \phi_* d\mu$$

(it can be seen from the definition that outer integration is subadditive, and that Fatou's lemma – in the present orientation – continues to be valid). □

In the remainder of this appendix I connect K-convergence in $\mathcal{R}_S(\mu)$ with the weak convergence topology for Young measures [9,2,7], which is defined as follows: A sequence (or generalised sequence) (δ_k) in $\mathcal{R}_S(\mu)$ is said to converge **weakly** (or **narrowly**) to a Young measure δ_0 (notation: $\delta_k \Longrightarrow \delta_0$) if

$$\liminf_{k\to\infty} I_g^*(\delta_k) \geq I_g^*(\delta_0)$$

for every $g : \Omega \times S \to [0, +\infty]$ such that $g(\omega, \cdot)$ is l.s.c. on S for a.e. $\omega \in \Omega$. (See also (See also [4, Thm.2.2] for some alternative, equivalent definitions.)

PROPOSITION A.8: *(a) Suppose that $\delta_k \xrightarrow{K} \delta_0$ in $\mathcal{R}_S(\mu)$. Then $\delta_k \Longrightarrow \delta_0$. (b) Suppose that $\delta_k \Longrightarrow \delta_0$ and that (δ_k) is tight.* Then there exists a subsequence (m) of (k) such that $\delta_m \xrightarrow{K} \delta_0$.*

Proof: (a) Immediate by Lemma 4.7(b). By Theorem A.5 there exist a subsequence (m) of (k) and $\delta_* \in \mathcal{R}_S(\mu)$ such that $\delta_m \xrightarrow{K} \delta_*$. Since $\delta_m \Longrightarrow \delta_0$, one finds by an easy application of the dominated convergence theorem on one side and an apllication of the weak convergence definition on the other, that $I_g(\delta_*) = I_g(\delta_0)$ for all g of the form $g(\omega, s) = \phi(\omega)c(s), \phi \in \mathcal{L}^1_{\mathbb{R}}(\mu), c \in \mathcal{C}_b(S)$. By use of the separating subset (c_j) once again, one proves easily that $\delta_*(\omega) = \delta_0(\omega)$ a.e. □

*The latter provision is automatically fulfilled when S is a Polish space [2, Example 2.5].

As an immediate consequence of Lemma A.7 and Proposition A.8 one has the following (see [2] for a different proof of a similar result):

PROPOSITION A.9: *Suppose that $\delta_k \Longrightarrow \delta_0$ in $\mathcal{R}_S(\mu)$ and suppose that (δ_k) is tight. Then*

$$\liminf_{k\to\infty} I_g^*(\delta_k) \geq I_g^*(\delta_0)$$

for every function $g : \Omega \times S \to (-\infty, +\infty]$ such that for a.e. ω,

$$g(\omega, \cdot) : S \to (-\infty, +\infty] \text{ is l.s.c. at every point of } \operatorname{supp} \delta_0(\omega)$$

relative to $\cup_{k=0}^{\infty} \operatorname{supp}(\delta_k(\omega))$,

$$g(\omega, \cdot) : S \to (-\infty, +\infty] \quad \text{is measurable on } S,$$

and such that for all $k \in \mathbf{N}$

$$g(\omega, s) \geq \phi_k(\omega) \quad \text{for all } s \in \operatorname{supp} \delta_k(\omega)$$

for some uniformly integrable sequence $(\phi_k) \subset \mathcal{L}^1_{\mathbf{R}}(\mu)$.

Proof: For g as given, let α denote the above limes inferior. There exists a subsequence (l) of (k) such that $\alpha = \lim_l I_g^*(\delta_l)$. Then, by Theorem A.8, there exists a a subsequence (m) of (l) such that $\delta_m \xrightarrow{K} \delta_0$. So Lemma A.7 gives $\alpha \geq I_g^*(\delta_0)$. □

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